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The Asymmetric Contact Process on a Finite Set

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The contact process on Z has one phase transition; let λ_c be the critical value at which the transition occurs. Let σ_N be the extinction time of the contact process on $\{0,..., N\}$. Durrett and Liu (1988), Durrett and Schonmann (1988), and Durrett, Schonmann, and Tanaka (1989) have respectively proved that the subcritical, supercritical, and critical phases can be characterized using a large finite system (instead of Z) in the following way. There are constants $\gamma_1(\lambda)$ and $\gamma_2(\lambda)$ such that if $\lambda < \lambda_c$, $\lim_{N\to\infty} \sigma_N/\log N = 1/\gamma_1(\lambda)$; if $\lambda > \lambda_c$, $\lim_{N\to\infty} \log \sigma_N/N = \gamma_2(\lambda)$; if $\lambda = \lambda_c$, $\lim_{N\to\infty} \sigma_N/N = \infty$ and $\lim_{N\to\infty} \sigma_N/N^4 = 0$ in probability. In this paper we consider the asymmetric contact process on Z when it has two distinct critical values $\lambda_{c1} < \lambda_{c2}$. The arguments of Durrett and Liu and of Durrett and Schonmann hold for $\lambda < \lambda_{c1}$ and $\lambda > \lambda_{c2}$. We show that for $\lambda \in [\lambda_{c1}, \lambda_{c2})$, $\lim_{N\to\infty} \sigma_N/N = -1/\alpha_l$ (where α_l is an edge speed) and for $\lambda = \lambda_{c2}$, $\lim_{N\to\infty} \log \sigma_N/\log N = 2$ in probability.

KEY WORDS: Asymmetric contact process; multiple phase transitions.

1. INTRODUCTION

The asymmetric contact process is a Markov process whose state at time t is denoted by ξ_t and which evolves according to the following rules. If there is a particle at $x \in Z$, then this particle gives birth to a new particle at $x+1$ at rate λ_r and at $x-1$ at rate λ_l, respectively. If a birth occurs at an already occupied site, then the birth is suppressed, so that at all times there is at most one particle per site. Finally, a particle dies at rate one. For $\lambda_r = \lambda_l$, this is the basic contact process, which has extensively been studied (see, for instance, Durrett[3]).

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Schonmann[13] studied the asymmetric contact process on Z. We will now introduce his notation and describe some of his results. Let $\lambda = (\lambda_r + \lambda_l)/2$ and for $\theta \in [0, \pi/2]$ let

$$c(\theta) = 2\frac{\cos\theta}{\sin\theta + \cos\theta} \qquad \text{and} \qquad s(\theta) = 2\frac{\sin\theta}{\sin\theta + \cos\theta}$$

We write (λ_l, λ_r) as

$$\lambda_r = \lambda c(\theta), \qquad \lambda_l = \lambda s(\theta)$$

Let ξ_t^0 be the asymmetric contact process whose initial state has only one particle located at the origin; we write $\xi_t^0(x) = 1$ if there is a particle at x at time t, otherwise $\xi_t^0(x) = 0$. The first critical line is defined by

$$\lambda_{c1}(\theta) = \sup\left\{\lambda\colon P\left(\sum_{x \in Z} \xi_t^0(x) \geqslant 1, \forall t > 0\right) = 0\right\}$$

and the second critical line is defined by

$$\lambda_{c2}(\theta) = \sup\{\lambda\colon \limsup_{t \to \infty} P(\xi_t^0(0) = 1) = 0\}$$

In words, the first critical value corresponds to the global survival of the process and the second critical value corresponds to the local survival. Note that $\lambda_{c1}(\theta) \leqslant \lambda_{c2}(\theta)$. For the basic contact process ($\theta = \pi/4$) it is known that the two critical values coincide: $\lambda_{c1}(\pi/4) = \lambda_{c2}(\pi/4)$. Schonmann[13] has shown that at least for some $\theta \in (0, \pi/2)$, $\lambda_{c1}(\theta) < \lambda_{c2}(\theta)$; he has also conjectured that for any $\theta \neq \pi/4$ the same strict inequality should hold. Note that this conjecture has been proved for branching random walks.[12]

In this paper we are interested in the problem of capturing the different phases of the asymmetric contact process when $\lambda_{c1}(\theta) \neq \lambda_{c2}(\theta)$ by looking at the evolution of this process on a large finite set (instead of Z). More precisely, we denote by η_t^N the asymmetric contact process on $\{0, 1,..., N\}$ and the evolution is the same as for ξ_t except that births on -1 and $N+1$ are suppressed. The initial state of η_t^N consists of one particle on each site of $\{0, 1,..., N\}$. Let

$$\sigma_N = \inf\left\{t > 0\colon \sum_{x=0}^{N} \eta_t^N(x) = 0\right\}$$

be the extinction time of the process. Since η_t^N is a finite Markov process with an absorbing state, σ_N is finite almost surely for any value of λ. Even so it is possible to characterize the different phases of the system in the following sense. Durrett and Liu[5] have proved for the basic contact that for any $\lambda < \lambda_{c1}(\pi/4) = \lambda_{c2}(\pi/4)$ there is a constant $\gamma_1(\lambda) \in (0, \infty)$ such that

$$\lim_{N \to \infty} \frac{\sigma_N}{\log N} = \frac{1}{\gamma_1(\lambda)} \quad \text{in probability} \tag{1.1}$$

On the other hand, Durrett and Schonmann[6] have proved for the basic contact process that for any $\lambda > \lambda_{c1}(\pi/4) = \lambda_{c2}(\pi/4)$ there is a constant $\gamma_2(\lambda) \in (0, \infty)$ such that

$$\lim_{N \to \infty} \frac{\log \sigma_N}{N} = \gamma_2(\lambda) \quad \text{in probability} \tag{1.2}$$

Finally, Durrett *et al.*[7] have proved for the basic contact process at the critical value $\lambda = \lambda_{c1}(\pi/4) = \lambda_{c2}(\pi/4)$

$$\lim_{N \to \infty} \frac{\sigma_N}{N} = \infty \qquad \text{and} \qquad \lim_{N \to \infty} \frac{\sigma_N}{N^4} = 0 \quad \text{in probability} \tag{1.3}$$

It is not difficult to see that the proofs of Durrett and Liu[5] and Durrett and Schonmann[6] can be adapted to the asymmetric contact process to prove that (1.1) holds for all $\lambda < \lambda_{c1}(\theta)$ and that (1.2) holds for all $\lambda > \lambda_{c2}(\theta)$, for all $\theta \in [0, \pi/2]$. The question that we address in this paper is: what is the order of magnitude of σ_N when $\lambda_{c1}(\theta) \neq \lambda_{c2}(\theta)$ and $\lambda \in [\lambda_{c1}(\theta), \lambda_{c2}(\theta)]$. In order to give a precise answer to this question, we need to introduce the following edge processes. Let ξ_t^{Z-} and ξ_t^{Z+} be asymmetric contact processes with starting configurations consisting of one particle at each negative integer and of one particle at each positive integer, respectively. Let the rightmost particle and the leftmost particle at time t be respectively defined by

$$r_t = \sup\{x \in Z : \xi_t^{Z-}(x) = 1\}, \qquad l_t = \inf\{x \in Z : \xi_t^{Z+}(x) = 1\}$$

As a consequence of the subadditive ergodic theorem (see Liggett,[10] Theorem 2.6, Chapter 6) there exist α_1 and α_2 such that[13]

$$\lim_{t \to \infty} \frac{r_t}{t} = \alpha_1(\lambda_r, \lambda_l) \in [-\infty, \infty) \quad \text{almost surely} \tag{1.4}$$

and

$$\lim_{t \to \infty} \frac{l_t}{t} = -\alpha_2(\lambda_r, \lambda_l) \in (-\infty, \infty] \quad \text{almost surely} \tag{1.5}$$

Moreover, Schonmann[13] has proved the following characterization of the first critical line (see Theorem 6):

$$\lambda_{c1}(\theta)=\sup\{\lambda\colon \alpha_1(\lambda_r, \lambda_l)+\alpha_2(\lambda_r, \lambda_l)\leqslant 0\} \tag{1.6}$$

and it is easy to see using Theorem 7 in Schonmann[13] that the following characterization of the second critical line holds:

$$\lambda_{c2}(\theta)=\sup\{\lambda\colon \min(\alpha_1(\lambda_r, \lambda_l), \alpha_2\lambda_r, \lambda_l))\leqslant 0\} \tag{1.7}$$

Schonmann (ref. 13, Theorem 10 and Corollary 4) has also proved that

$$\alpha_1(\lambda_r, \lambda_l)+\alpha_2(\lambda_r, \lambda_l)=0$$

if and only if $\lambda=\lambda_{c1}(\theta)$, and

$$\min(\alpha_1(\lambda_r, \lambda_l), \alpha_2(\lambda_r, \lambda_l))=0$$

if and only if $\lambda=\lambda_{c2}(\theta)$. Therefore, for $\lambda\in[\lambda_{c1}(\theta), \lambda_{c2}(\theta))$ we must have

$$\alpha_1(\lambda_r, \lambda_l)\,\alpha_2(\lambda_r, \lambda_l)<0$$

We are now ready to state our results.

Theorem 1. For $\lambda\in[\lambda_{c1}(\theta), \lambda_{c2}(\theta))$ we have $\alpha_1(\lambda_r, \lambda_l)\,\alpha_2(\lambda_r, \lambda_l)<0$. If $\alpha_i(\lambda_r, \lambda_l)<0$ ($i=1$ or 2), then

$$\lim_{N\to\infty}\frac{\sigma_N}{N}=-\frac{1}{\alpha_i(\lambda_r, \lambda_l)}\quad \text{in probability}$$

We conjecture that the result in Theorem 1 holds in more general situations in the sense that when there are two phase transitions then σ_N has an order of magnitude equal to the "radius" of the finite set. For instance, we believe that an asymmetric contact process on a finite set with N sites in Z^d has an extinction time with an order of magnitude equal to $N^{1/d}$ between the two phase transitions. For a symmetric contact process on a homogeneous tree with N sites we believe that the extinction time is of order $\log N$ between the two phase transitions.[11]

Now we turn to the behavior of σ_N at the second critical value.

Theorem 2. Assume that $\lambda_{c1}(\theta)\neq\lambda_{c2}(\theta)$; then at $\lambda=\lambda_{c2}(\theta)$ we have for any $\varepsilon>0$ and any sequence K_N going to infinity with N

$$\lim_{N\to\infty}\frac{\sigma_N}{K_N N^2}=0 \qquad \text{and} \qquad \lim_{N\to\infty}\frac{\sigma_N}{N^2(\log N)^{-1-\varepsilon}}=\infty \quad \text{in probability}$$

In particular,

$$\lim_{N \to \infty} \frac{\log \sigma_N}{\log N} = 2 \quad \text{in probability}$$

We conjecture that σ_N/N^2 converges in distribution but not in probability. Comparing (1.3) and Theorem 2, we see that we get in the critical asymmetric cases results which are much more precise than in the critical symmetric case. The asymmetric case is easier to deal with because at $\lambda = \lambda_{c1}(\theta)$ the edge speeds are not zero and at $\lambda = \lambda_{c2}(\theta)$ one edge speed is zero but the fluctuations of the edge are Brownian,[8,9] while at $\lambda = \lambda_c(\pi/4)$ the fluctuations are not known (but are believed to be non-Brownian) and the edge speeds are zero.

At $\lambda = \lambda_{c2}(\theta)$ we have $\min(\alpha_1(\lambda_r, \lambda_l), \alpha_2(\lambda_r, \lambda_l)) = 0$ and $\alpha_1(\lambda_r, \lambda_l) + \alpha_2(\lambda_r, \lambda_l) > 0$. To fix the notation, assume that $\alpha_1(\lambda_r, \lambda_l) = 0$. One of the keys in our proof of Theorem 2 is the following consequence of a representation theorem for the edge due to Kuczek.[9]

Theorem 3. At $\lambda = \lambda_{c2}(\theta) \neq \lambda_{c1}(\theta)$, if $\lim_{t \to \infty} r_t/t = \alpha_1(\lambda_r, \lambda_l) = 0$, then

$$\lim_{t \to \infty} \frac{r_t}{t^{1/2}(\log t)^{1/2+\varepsilon}} = 0 \quad \text{almost surely}$$

for all $\varepsilon > 0$.

2. CONSTRUCTION AND PROOF OF THEOREM 1

We begin by recalling the graphical construction of the asymmetric contact process (for more details, see Durrett[2]). We associate each site $x \in Z$ with three independent Poisson processes: $\{T_n^{x,x-1}: n \geq 1\}$ has rate λ_l, $\{T_n^{x,x+1}: n \geq 1\}$ has rate λ_r, and $\{T_n^x: n \geq 1\}$ has rate 1. We make these Poisson processes independent from site to site. For each $x \in Z$ and $n \geq 1$ we write a δ mark at the point (x, T_n^x) while we draw arrows from $(x, T_n^{x,x+1})$ to $(x+1, T_n^{x,x+1})$ and form $(x, T_n^{x,x-1})$ to $(x-1, T_n^{x,x-1})$. We say that there is a path from (x, s) to (y, t) if there is a sequence of times $s_0 = s < s_1 < s_2 < \cdots < s_{n+1} = t$ and spatial locations $x_0 = x, x_1, ..., x_n = y$ so that for $i = 1, 2, ..., n$ there is an arrow from x_{i-1} to x_i at time s_i and the vertical segments $\{x_i\} \times (s_i, s_{i+1})$ for $i = 0, ..., n$ do not contain any δ. We denote the event "there is a path from (x, s) to (y, t)" by $\{(x, s) \to (y, t)\}$. To construct the contact process if the set of occupied sites at the initial time is A, we let $\xi_t^A(x) = 1$ if there is a path from $(y, 0)$ to (x, t) for some $y \in A$.

We use this construction for the asymmetric contact process on a finite set as well, but we suppress the arrows which have an endpoint outside of the finite set.

Proof of Theorem 1. We will simplify the notation and write α_i instead of $\alpha_i(\lambda_r, \lambda_l)$ whenever no confusion is possible. To fix the notation, we will prove Theorem 1 assuming that $\alpha_1 < 0$ and $\alpha_2 > 0$ so that the leftmost and rightmost particles drift to the left. We will also denote by C, γ, c_1, c_2,... constants in $(0, \infty)$.

Fix $\varepsilon > 0$; we have that

$$P\left(\sigma_N > \left(\frac{-1}{\alpha_1} + \varepsilon\right) N\right) \leqslant P(r^N_{(-1/\alpha_1 + \varepsilon)N} > 0) \tag{2.1}$$

where r_t^N is the rightmost particle of η_t^N. But we can couple η_t^N and ξ_t in such a way that conditioned on the survival of η_t^N we have (see Liggett,[10] Theorem 2.2, Chapter 6)

$$r_t^N = r_t + N \tag{2.2}$$

From (2.1) and (2.2) we get

$$P\left(\sigma_N > \left(\frac{-1}{\alpha_1} + \varepsilon\right) N\right) \leqslant P(r_{(-1/\alpha_1 + \varepsilon)N} > -N) \tag{2.3}$$

But since r_t/t converges to α_1 as t goes to infinity, the RHS of (2.3) converges to 0 as N goes to infinity.

We now turn to the other bound. We adapt the ideas of Durrett *et al.*[7] [see (3.1) there] to our case. Let r'_t be the right edge of the asymmetric contact process on Z, starting with one particle on each site of $(-\infty, N - \log N)$. Recall that $\alpha_1 < 0$ and consider the following event:

$$G = \{r'_t \in (-N\alpha_1 \varepsilon/2, N) \text{ for all } t \in [0, (-1/\alpha_1 - \varepsilon) N]\}$$

Lemma 1. $\lim_{N \to \infty} P(G) = 1$.

Proof of Lemma 1. By translation invariance

$$P(\exists t \in [0, (-1/\alpha_1 - \varepsilon) N]: r'_t \geqslant N) = P(\exists t \in [0, (-1/\alpha_1 - \varepsilon) N]: r_t \geqslant \log N) \tag{2.4}$$

The RHS of (2.4) is less than

$$P(\exists t \in [0, c \log N]: r_t \geqslant \log N) + P(\exists t > c \log N: r_t > 0) \tag{2.5}$$

Since r_t is dominated by a Poisson process with rate λ_r, we can pick c small enough so that the first term in (2.5) goes to zero as N goes to infinity. The second term in (2.5) also converges to zero since r_t/t goes to $\alpha_1 < 0$ almost surely as t goes to infinity.

We now consider

$$P(\exists t \in [0, (-1/\alpha_1 - \varepsilon) N]: r'_t \leqslant -N\alpha_1 \varepsilon/2)$$
$$= P(\exists t \in [0, (-1/\alpha_1 - \varepsilon) N]: r_t \leqslant -N\alpha_1 \varepsilon/2 - N + \log N) \quad (2.6)$$

(2.6) is less than

$$P(\exists t < c_1 N: r_t \leqslant -N\alpha_1 \varepsilon/2 - N + \log N)$$
$$+ P(\exists t \in [c_1 N, (-1/\alpha_1 - \varepsilon) N]: r_t \leqslant -N\alpha_1 \varepsilon/2 - N + \log N) \quad (2.7)$$

Again taking c_1 small enough and comparing r_t with a Poisson process shows that the first term in (2.7) goes to 0 as N goes to infinity. Note that if $t \leqslant (-1/\alpha_1 - \varepsilon) N$ and if c_2 is in $(0, \alpha_1 \varepsilon/2/(1/\alpha_1 + \varepsilon))$, then $(\alpha_1 - c_2)\, t \geqslant -N\alpha_1 \varepsilon/2 - N + \log N$ and therefore the second term in (2.7) is less than

$$P(\exists t > c_1 N: r_t \leqslant (\alpha_1 - c_2)\, t)$$

and this term goes to zero as N goes to infinity. This completes the proof of Lemma 1. ■

To show that $\lim_{N \to \infty} P(G) = 1$ implies that $\lim_{N \to \infty} P(\sigma_N > N(-1/\alpha_1 - \varepsilon)) = 1$, we need the following properties of the asymmetric contact process. The dual of the asymmetric contact process $\xi_{\lambda_r, \lambda_l}(t)$ is $\xi_{\lambda_l, \lambda_r}(t)$ in the sense that if A and B are subsets of Z, then

$$P(\xi^A_{\lambda_r, \lambda_l}(t)(x) = 0, \forall x \in B) = P(\xi^B_{\lambda_l, \lambda_r}(t)(x) = 0, \forall x \in A) \quad (2.8)$$

We will also need that the edge

$$r_{\lambda_r, \lambda_l}(t) \quad \text{has the same distribution as} \quad -l_{\lambda_l, \lambda_r}(t) \quad (2.9)$$

We have that

$$P(\sigma_N < N(-1/\alpha_1 - \varepsilon)) \leqslant P(G^c) + P(G; \sigma_N < N(-1/\alpha_1 - \varepsilon)) \quad (2.10)$$

On G there is a path from $(-\infty, N - \log N) \times \{0\}$ to $[-N\alpha_1 \varepsilon/2, \infty) \times \{N(-1/\alpha_1 - \varepsilon)\}$ which does not touch the line $\{N\} \times [0, N(-1/\alpha_1 - \varepsilon)]$. This path exists for the contact process restricted to $\{0, 1,..., N\}$ if it does not touch the line $\{0\} \times [0, N(-1/\alpha_1 - \varepsilon)]$. By (2.8) observe that the probability of having a path touch $\{0\} \times [0, N(-1/\alpha_1 - \varepsilon)]$ and end up in $(-N\alpha_1 \varepsilon/2, \infty)$ is the same as the probability of a path in the dual from $(-N\alpha_1 \varepsilon/2, \infty) \times \{0\}$ to $\{0\} \times [0, N(-1/\alpha_1 - \varepsilon)]$. Using (2.9), we see that such a path in the dual has the same probability as the event $\{\exists t \in [0, N(-1/\alpha_1 - \varepsilon)]: r_t > -N\alpha_1 \varepsilon/2\}$. So the RHS of (2.10) is less than

$$P(G^c) + P(G)\, P(\exists t \in [0, N(-1/\alpha_1 - \varepsilon)]: r_t > -N\alpha_1 \varepsilon/2) \qquad (2.11)$$

By an argument similar to the one used to prove Lemma 1, the second term in the RHS of (2.11) goes to zero as N goes to infinity. Using Lemma 1 for the first term in the RHS of (2.11), we get

$$\lim_{N \to \infty} P(\sigma_N > N(-1/\alpha_1 - \varepsilon)) = 1$$

This completes the proof of Theorem 1. ■

3. PROOF OF THEOREM 3

Kuczek[9] has proved that there is a renewal process $N(t)$ with interrenewal times τ_i, $i = 1, 2,...,$ and for which the last renewal time before time t is denoted by $S_{N(t)} = \sum_{i=1}^{N(t)} \tau_i$ such that the edge at time $S_{N(t)}$ can be represented by

$$r_{S_{N(t)}} = \sum_{i=1}^{N(t)} X_i \qquad (3.1)$$

where the random vectors (X_i, τ_i), $i = 1, 2,...,$ are independent, identically distributed with all moments. The proof of Kuczek is written in discrete time for the supercritical symmetric contact process, but can be adapted with no difficulties to the asymmetric contact process with $\lambda > \lambda_{c1}(\theta)$.

We have

$$r_t = r_{S_{N(t)}} + r_t - r_{S_{N(t)}} \qquad (3.2)$$

It is easy to see that $\alpha_1 = 0$ implies that $E(X_1) = 0$. Now, a classical law of large numbers (see, for instance, Theorem 8.2, Chapter 1 in ref. 4) implies that since the X_i are i.i.d. with finite variance,

$$\lim_{n \to \infty} \frac{1}{n^{1/2}(\log n)^{1/2+\varepsilon}} \sum_{i=1}^{n} X_i = 0 \quad \text{almost surely} \qquad (3.3)$$

Since $\lim_{t \to \infty} N(t)/t = 1/E(\tau_1)$ almost surely, (3.1) and (3.3) imply that

$$\lim_{t \to \infty} \frac{r_{S_{N(t)}}}{t^{1/2}(\log t)^{1/2+\varepsilon}} = 0 \quad \text{almost surely} \qquad (3.4)$$

So in order to prove Theorem 3, we need to control the other terms in the RHS of (3.2) using the following lemma.

Lemma 2. For all $b > 0$,

$$\lim_{t \to \infty} \frac{t - S_{N(t)}}{t^b} = 0 \quad \text{almost surely}$$

Proof of Lemma 2. This proof uses the method indicated in ref. 4 (Exercise 4.13, p. 107) with higher moments. Fix $a > 0$,

$$P(t - S_{N(t)} > at^b) \leqslant P(\tau_{N(t)+1} > at^b) \leqslant P(\max_{1 \leqslant i \leqslant N(t)+1} \{\tau_i\} > at^b) \tag{3.5}$$

The RHS of (3.5) is less than

$$P(N(t) > t^k) + t^k P(\tau_1 > at^b) \tag{3.6}$$

where $k > 1$ is a constant. We have

$$P(N(t) > t^k) \leqslant P\left(\sum_{i=1}^{t^k} \tau_i < t\right) \leqslant \frac{\mathrm{Var}(\sum_{i=1}^{t^k} \tau_i)}{[t^k E(\tau_1) - t]^2} \tag{3.7}$$

where the last inequality is Chebyshev's inequality. From (3.7) we see that

$$P(N(t) > t^k) \leqslant \frac{c}{t^k} \tag{3.8}$$

for a constant $c > 0$.

We now turn to the second term in (3.6). Using that $E(\tau_1^p) < \infty$ for all $p > 0$, we have that

$$\sup_t t^p P(\tau_1 > t) \leqslant E(\tau_1^p) < \infty \tag{3.9}$$

(3.9) implies that

$$t^k P(\tau_1 > at^b) \leqslant E(\tau_1^p)\, a^{-p} t^{k-bp} \tag{3.10}$$

We pick p large enough in order to have $k - bp < -1$. Using (3.10) and (3.8), we see that there are constants $c_1 > 0$ and $k_1 > 1$ such that

$$P(t - S_{N(t)} > at^b) \leqslant \frac{c_1}{t^{k_1}} \tag{3.11}$$

A Borel–Cantelli argument completes the proof of Lemma 2. ■

We are now ready to prove Theorem 3. Consider

$$P(|r_t - r_{S_{N(t)}}| > at^b) \leqslant P(t - S_{N(t)} > t^{b/2}) + P(|r_t - r_{S_{N(t)}}| > at^b,\ t - S_{N(t)} < t^{b/2}) \quad (3.12)$$

But r_t is dominated by a Poisson process with rate λ_r; therefore if we denote a random variable which has a Poisson distribution with mean c by $\mathscr{P}(c)$, the second term in the RHS of (3.12) is less than

$$P(\mathscr{P}(\lambda_r t^{b/2}) > at^b) \leqslant \exp(-c_2 t^{b/2}) \quad (3.13)$$

where $c_2 > 0$ is a constant and the inequality is an exponential Chebyshev inequality. We also use (3.11) to bound the first term on the RHS of (3.12) to get

$$P(|r_t - r_{S_{N(t)}}| > at^b) \leqslant \frac{c_3}{t^{k_1}}$$

for a constant c_3. A Borel–Cantelli argument shows that

$$\lim_{t \to \infty} \frac{|r_t - r_{S_{N(t)}}|}{t^b} = 0 \quad \text{almost surely} \quad (3.14)$$

(3.4) together with (3.14) prove Theorem 3. ■

4. PROOF OF THEOREM 2

We first prove the lower bound in Theorem 2:

$$\lim_{N \to \infty} \frac{\sigma_N}{N^2(\log N)^{-1-\varepsilon}} = \infty \quad \text{in probability} \quad (4.1)$$

To do so, we use Theorem 3 together with an argument analogous to the one used for Theorem 1. Let r_t^1 be the right edge of the asymmetric contact process on Z, starting with one particle on each site of $(-\infty, 2N/3)$. Fix $A > 0$ and $\varepsilon > 0$; consider the following event:

$$G^1 = \{r_t^1 \in (N/3, N) \text{ for all } t \in [0, AN^2(\log N)^{-1-\varepsilon}]\}$$

Similarly to Lemma 1, we prove that

$$\lim_{N \to \infty} P(G^1) = 1 \quad (4.2)$$

Note that by translation invariance

$$P((G^1)^c) = P(\exists t \in [0, AN^2(\log N)^{-1-\varepsilon}]: |r_t| > N/3) \tag{4.3}$$

For a constant c small enough, the probability that $|r_t| > N/3$ for some $t < cN$ goes to zero as N goes to infinity (compare with a Poisson process). So to control (4.3) we only need to consider

$$P((G^1)^c) = P(\exists t \in [cN, AN^2(\log N)^{-1-\varepsilon}]: |r_t| > N/3) \tag{4.4}$$

A little computation shows that if $t < AN^2(\log N)^{-1-\varepsilon}$, then $N > t^{1/2}(\log t)^{1/2+\varepsilon/3}/A$. Therefore (4.4) is less than

$$P\left(\exists t > cN: r_t > \frac{t^{1/2}(\log t)^{1/2+\varepsilon/3}}{3A}\right) \tag{4.5}$$

By Theorem 3, (4.5) goes to zero as N goes to infinity. And this completes the proof of (4.2). Using the dual process in an argument similar to the one used after Lemma 1, (4.2) implies that for all $A > 0$

$$\lim_{N \to \infty} P(\sigma_N > AN^2(\log N)^{-1-\varepsilon}) = 1$$

and this implies (4.1).

We now turn to the upper bound: assume K_N is a sequence going to infinity with N; we want to prove that

$$\lim_{N \to \infty} \frac{\sigma_N}{K_N N^2} = 0 \text{ in probability} \tag{4.6}$$

Observe that if we couple the finite and the infinite systems

$$P(\sigma_N > K_N N^2) \leqslant P(\inf_{0 \leqslant t \leqslant K_N N^2} r'_t \geqslant 0) \tag{4.7}$$

where r'_t is the right edge of the asymmetric contact process on Z, starting with one particle on each site of $(-\infty, N]$. Using translation invariance, we get

$$P(\sigma_N > K_N N^2) \leqslant P(\inf_{0 \leqslant t \leqslant K_N N^2} r_t \geqslant -N) \tag{4.8}$$

Galves and Presutti[8] proved that the process

$\varepsilon r_{\varepsilon^{-2}t}$ converges in law, when $\varepsilon \to 0$, to a Brownian motion B_t

In particular, if $\varepsilon = (K_N N^2)^{-1/2}$, we get

$$\lim_{N \to \infty} P\left(\inf_{0 \leqslant t \leqslant 1} \frac{r_{KN^2 t}}{(K_N N^2)^{1/2}} \geqslant \frac{-N}{(K_N N^2)^{1/2}} \right) = P(\inf_{0 \leqslant t \leqslant 1} B_t \geqslant 0) = 0 \quad (4.9)$$

(4.9) together with (4.8) prove (4.6). ■

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